On the Possible Violation of Sum Rules for Higher-Twist Parton Distributions

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Precise measurements of polarized electro-production and Drell-Yan in the deep inelastic limit will soon provide first information on the higher twist parton distributions $g_T(x)$ and $h_L(x)$. Sum rules for higher-twist structure functions are only valid provided the corresponding Compton amplitudes satisfy un-subtracted dispersion relations. Subtracted dispersion relations have to be used when the (real part of the) forward scattering amplitudes does not fall off rapidly enough for $\nu \rightarrow \infty$ (fixed Q^2). Formally, such subtractions lead to δ-functions at the origin in the parton distributions, which are not accessible to experiment, and the integral over the data fails to satisfy the sum rule. The δ-functions in the parton distributions can be identified with the zero-modes that appear in light-front quantization. An explicit *infinite momentum boost* identifies these soft quark modes with low momentum contributions to the self-energy.

I. INTRODUCTION

With the advent of precise deep inelastic scattering (DIS) experiments it will soon become possible to measure the proton's polarized higher-twist structure functions $g_T(x)$ and $h_L(x)$. $g_T(x) \equiv g_2(x) - g_1(x)$ is obtained in polarized electro-production [1], where one finds for the parallel-anti-parallel asymmetry

$$\frac{d\sigma^{\uparrow\uparrow}}{dq^2dE'} - \frac{d\sigma^{\uparrow\downarrow}}{dq^2dE'} = \frac{4\pi\alpha}{E^2q^2}\left[\left(E + E'\cos\theta\right)MG_1 + q^2G_2\right], \tag{1.1}$$

where E, E' are the lab energy of the initial/final lepton, $\uparrow$, $\downarrow$ denote nucleon/lepton felicities and θ is the lab angle of the final lepton. In the parton model, νG_1 scales while only $\nu^2 G_2$ has a finite scaling limit as $Q^2 \equiv -q^2 \rightarrow \infty$, $\nu \equiv E - E' \rightarrow \infty$ and $x_{Bj} \equiv Q^2/2M\nu$ fixed (the Bjorken limit). Using the operator product expansion one finds (modulo logarithmic scaling violations) $M^2\nu G_1 \stackrel{Bj}{\longrightarrow} \sum_q e_q^2 g_1^q(x_{Bj}) + g_1^q(-x_{Bj})$ and $M\nu^2 G_2 \stackrel{Bj}{\longrightarrow} \sum_q e_q^2 g_2^q(x_{Bj}) + g_2^q(-x_{Bj})$,[1] where the parton distributions g_1^q and g_2^q are defined through the light cone correlations [2]

$$\int \frac{d\lambda}{2\pi} e^{i\lambda x}\langle PS|\bar{q}(0)\gamma^\mu\gamma_5 q(\lambda n)|PS\rangle \tag{1.2}$$
$$= 2\left[g_1^q(x)p^\mu(S\cdot n) + g_T^q(x)S_T^\mu + M^2 g_3^q(x)n^\mu(S\cdot n)\right],$$

where $n^2 = p^2 = 0$, $n^+ = p^- = 0$, $P^\mu = p^\mu + \frac{M^2}{2}n^\mu$, $S^\mu = p^\mu(S\cdot n) + n^\mu(S\cdot p) + S_T^\mu$ and $g_T^q(x) = g_1^q(x) + g_2^q(x)$. The different scaling behavior of G_1 and G_2 as $p^+ \rightarrow \infty$ (Breit frame) is reflected in the coefficients in Eq.(1.2), where the coefficient of g_2^q is independent of p^+, while the contribution from g_1^q in Eq.(1.2) grows linearly with p^+. This is typical for operators of different twist. Clearly, the non-leading role played by G_2 makes it rather difficult to separate them from the leading twist term G_1 (and from $1/Q^2$ corrections to G_1). The HERMES experiment at DESY will be the first attempt for a precise determination of $g_2^q(x)$ [3], which has received special attention due to the existence of the Burkhardt-Cottingham sum rule [4]

$$\int_{-1}^1 dx g_2^q(x) \equiv \int_0^1 dx\left[g_2^q(x) + g_2^{\bar{q}}(x)\right] = 0, \tag{1.3}$$

which seems to follow easily from the OPE result (1.2) [5,6]. To see this, one can integrate Eq.(1.2) over x [using the fact that the r.h.s. has support only for $x \in (-1;1)$], where we find

$$\langle PS|\bar{q}(0)\gamma^\mu\gamma_5 q(0)|PS\rangle = \tag{1.4}$$
$$2\int_{-1}^1\left[g_1^q(x)p^\mu(S\cdot n) + g_T^q(x)S_T^\mu + M^2 g_3^q(x)n^\mu(S\cdot n)\right].$$

from rotational invariance it follows that the l.h.s. in Eq.(1.4) is proportional to the spin vector S_μ and thus the $g_i^q(x)$ must satisfy

$$\int_{-1}^1 dx g_1^q(x) = \int_{-1}^1 dx g_T^q(x)$$
$$\int_{-1}^1 dx g_1^q(x) = 2\int_{-1}^1 dx g_3^q(x). \tag{1.5}$$

The first of these conditions (1.5) is abovementioned Burkhardt-Cottingham sum-rule, while the second is a (probably useless) sum-rule for g_3.[2]

[1] $g_i^q(-x)$ is sometimes denoted $g_i^{\bar{q}}(x)$ in the literature.

[2] Useless, because no experiment is known that could access g_3 with sufficient precision and because Regge arguments suggest that the sum rule is highly divergent.

The way I presented the derivation of these sum rules, it seems that they are mere consequences of rotational invariance and hence there should be no question about their validity in QCD. However, there are several subtleties if one wants to turn these sum rules for the g_i's (the light-cone correlators) into sum rules for the G_i's (the experimentally measured cross sections). Discussing these subtleties will be the main purpose of the rest of this paper. Here I only mention that the OPE result is derived using dispersion relations for the lepton-nucleon forward scattering amplitude and in general there is always the possibility that the dispersion relation does not converge or that the dispersion relation requires a finite subtraction.

Similar results apply for the chirally odd parton distributions $h_i^q(x)$ defined as [2]

$$\begin{aligned} &\int \frac{d\lambda}{2\pi} e^{i\lambda x} \langle PS|\bar{q}(0)\sigma^{\mu\nu} i\gamma_5 q(\lambda n)|PS\rangle \\ &= 2\left[h_1^q(x)\left(S_T^\mu p^\nu - S_T^\nu p^\mu\right)/M \right. \\ &\quad + h_L^q(x) M \left(p^\mu n^\nu - p^\nu n^\mu\right)(S\cdot n) \\ &\quad \left. + h_3^q(x) M \left(S_T^\mu n^\nu - S_T^\nu n^\mu\right)\right], \end{aligned} \tag{1.6}$$

where $h_L^q(x) = h_1^q(x) + h_2^q(x)/2$. Since they are chirally odd, they cannot be measured in inclusive electroproduction. One promising way of measuring them is the polarized Drell-Yan process [7] (see also Ref. [8]): If both target and beam are longitudinally polarized, one finds for the spin assymetry [9]

$$A_{LL} = \frac{\sum_q e_q^2 g_1^q(x) g_1^q(-y)}{\sum_q e_q^2 f_1^q(x) f_1^q(-y)}. \tag{1.7}$$

If both target and beam are transversally polarized, [9]

$$A_{TT} = \frac{\sin^2\theta\cos 2\phi}{1+\cos^2\theta} \frac{\sum_q e_q^2 h_1^q(x) h_1^q(-y)}{\sum_q e_q^2 f_1^q(x) f_1^q(-y)} \tag{1.8}$$

and if one is longitudinal and the other transverse, [2,10]

$$A_{LT} = \frac{2\sin 2\theta\cos\phi}{1+\cos^2\theta} \frac{M}{\sqrt{Q^2}} \tag{1.9}$$

$$\frac{\sum_q e_q^2 \left(g_1^q(x) y g_T^q(-y) - x h_L^q(x) h_1^q(-y)\right)}{\sum_q e_q^2 f_1^q(x) f_1^q(-y)} \tag{1.10}$$

.

An alternative way of measuring h_1 is through semi-inclusive polarized electro-production [2,3].

Similarly to the sum rules for the g_i^q one can derive sum rules for the h_i^q by integrating Eq.(1.6) and one finds [5,6], using again rotational invariance,

$$\begin{aligned} \int_{-1}^1 dx h_1^q(x) &= \int_{-1}^1 dx h_L^q(x) \\ \int_{-1}^1 dx h_1^q(x) &= 2\int_{-1}^1 dx h_3^q(x). \end{aligned} \tag{1.11}$$

As is the case for the g_3-sum rule, the h_3-sum rule is most certainly useless, but since measurements for h_1 and h_L are under way, it will soon be possible to test the h_L-sum rule (1.11) experimentally. In principle, the same comments that I made above for the g_T sum rule also apply for the h_L sum rule. However, as I will discuss in Section III, it seems to be violated already in perturbation theory.

The paper is organized as follows. In Section II, I will explain the connection between the light-cone correlation functions and the experimentally measured quantities and why the formal sum rules — derived for the light-cone correlation functions — may be violated when applied to the measured structure functions. In Section III, I will show that the h_2 sum rule is violated in QCD. In the following sections I will study some model filed theories and simple (spin-independent) higher-twist parton distributions and I will attempt to shed some light on the physics of such violations of sum rules. In the Appendix, I will draw connections to the notorious zero-modes plaguing the light-front community.

II. OPERATOR PRODUCT EXPANSION, DISPERSION RELATIONS AND ALL THAT

Deep inelastic scattering (DIS) experiments are performed in the region $0 < Q^2 < 2M\nu$ [11]. The operator product expansion is an expansion around $Q^2 = \infty$ and must diverge for $Q^2 < 2M\nu$ because of the singularities of the hadronic tensor (the physical discontinuity along the cut). The consequences of these statements are easily demonstrated by considering some hadronic tensor $T(x, Q^2)$, where $x = Q^2/2M\nu$. To keep the discussion as general as possible, we will consider here some generic forward scattering amplitude, without specifying the current. This will allow us later to draw very general conclusions. A very similar discussion, for the specific example of $g_2(x)$, can be found in Ref. [12].

As an function of x, $T(x, Q^2)$ is analytic in the complex plane, cut along $0 < x < 1$. The discontinuity of the imaginary part along the cut is (up to trivial kinematic factors) the experimentally measured cross section, which we will denote by $G(x, Q^2)$. In order to apply the OPE, one needs to relate $T(x, Q^2)$ for $|x| > 1$ to $T(x, Q^2)$ for $|x| < 1$. From the theory of complex functions it is well known that an analytic function is determined by its singularities in the complex plane — up to a polynomial! This implies ($|x| > 1$)

$$T(x, Q^2) = p\left(\frac{1}{x}, Q^2\right) + \frac{1}{\pi}\int_0^1 dx' \frac{x}{x^2 - x'^2} G(x'), \tag{2.1}$$

where $p\left(\frac{1}{x}, Q^2\right)$ is a polynomial in $\frac{1}{x}$ whose coefficients do in general depend on Q^2. In the language of Regge theory, such polynomial subtractions are called “fixed poles” [13]. Note that we specialized here on an odd function

of $1/x$, the generalization to the even case follows analogously. Usually, only a few coefficients of the polynomial are allowed to be nonzero, since otherwise unitarity bounds are violated, but typically this places no restriction on the lowest few terms.

For $x > 1$ one can apply the OPE and one finds

$$T(x, Q^2) = \sum_{n=1,3,5,..} \frac{1}{x^n} a_n, \tag{2.2}$$

where the coefficients are related to matrix elements of local operators. If one allows for generalized functions, it is always possible to find a function $g(x')$, such that

$$a_n = \int_{-1}^{1} dx' x'^{n-1} g(x') \qquad \forall n = 1, 3, ... \tag{2.3}$$

In the case of the moments appearing in the OPE for DIS this function typically is a Fourier transform of a correlation function along the light-cone, such as the g_i and h_i introduced in Eqs.(1.2,1.6).

In order to demonstrate what happened if the subtraction polynomial does not vanish, let us study an example, where the polynomial is finite[3] and first order[4] $p\left(\frac{1}{x}, Q^2\right) = \frac{1}{x} p_1(Q^2)$. Upon inserting p in Eq.(2.1) and introducing the moments of the measured structure function

$$b_n \equiv \frac{1}{\pi} \int_0^1 dx' x'^{n-1} G(x') \tag{2.4}$$

one thus finds

$$\begin{aligned} a_n &= b_n \qquad n = 3, 5, .. \\ a_1 &= p_1 + b_1 \ . \end{aligned} \tag{2.5}$$

Such a result, if p_1 is nonzero, has a two major consequences:

- Any sum rule, derived for a_1, fails to be satisfied for b_1. Whether or not the sum rule derived using the OPE applies also to the experimentally observed structure functions depends on whether or not the amplitudes satisfy un-subtracted dispersion relation.

- Even when $G(x')$ is a smooth function, since all moments except the lowest moment of $G(x)/\pi$ and $g(x) + g(-x)$ are the same, it must be that $g(x)$ contains a δ-function at the origin (which obviously afflicts only the lowest moment)

$$g(x) + g(-x) = p_1 \delta(x) + \frac{1}{\pi} G(x) \tag{2.6}$$

A few more comments are in order here. So far, I have only discussed what happens if there is such a subtraction. In the rest of the paper I will show (studying QCD perturbation theory as well as some toy models) that this is actually a very common phenomenon. In these examples I will furthermore demonstrate, that the light-cone correlations indeed contain δ-functions, even when the "data" is very smooth, and that the coefficient of the δ-functions indeed coincides with subtraction constants in dispersion relations. Furthermore, I should emphasize that the δ-functions do **not** arise from some obscure $Q^2 \to \infty$ limit but from inverting a moment transformation at fixed and finite Q^2.

III. PERTURBATIVE ANALYSIS OF H_L AND G_T

So far, the discussion has been deliberately abstract. Even though we have seen what happens *if* the dispersion relations contain subtractions, we have not yet shown *whether* this actually happens. As a first step in this direction we will perform a perturbative analysis of $h_L(x)$ and $g_T(x)$ in this section. Let us first look at the $\mathcal{O}(g^2)$ corrections to $h_L(x)$ for a target that consists of a quark of momentum P and spin S, dressed by one gluon loop (Fig.1).

FIG. 1. One loop correction to $g_T(x)$ and $h_L(x)$ in QCD. The cross represents the insertion of the light-cone correlator.

For simplicity, we will assume $P_\perp = 0$, but the generalization to nonzero transverse momenta is straightforward. From the definition (1.6) one finds

[3] Note that in general it may happen that the integral in Eq.(2.1) does not converge and hence one needs an infinite subtraction. Such a case may be considered as a limit of the situation assumed here.

[4] Typically, higher orders are not allowed anyway because of unitarity constraints.

$$h_L(x)\frac{S^-}{M} = -g^2\bar{u}(P,S)\int \frac{d^4k}{(2\pi)^4}\delta(k^+ - xp^+)\gamma^\mu \frac{i}{\not{k} - m + i\varepsilon}\sigma^{+-}i\gamma_5 \frac{i}{\not{k} - m + i\varepsilon}\gamma^\mu u(P,S)D_{\mu\nu}(P-k), \tag{3.1}$$

where

$$D_{\mu\nu}(q) = \frac{-i}{q^2 + i\varepsilon}\left(g^{\mu\nu} - \frac{q^\mu n^\nu + q^\nu n^\mu}{nq}\right) \tag{3.2}$$

is the gluon propagator in the light-front gauge [5] and m is the current quark mass. The term $\delta(k^+ - xp^+)$ arises from the λ-integration in Eq.(1.6) which projects out quarks of momentum fraction x.
Straightforward application of the γ-matrix algebra yields (using $\not{P}u(P,S) = m\,u(P,S)$ and dropping terms odd under $\vec{k}_\perp \to -\vec{k}_\perp$)

$$h_L(x)\frac{S^-}{M} = g^2\bar{u}(P,S)\int \frac{d^4k}{(2\pi)^4}\frac{\delta(k^+ - xp^+)}{(p-k)^2 + i\varepsilon}\left\{\frac{-4m\,(k^+\gamma^- - k^-\gamma^+)\,i\gamma_5}{(k^2 - m^2 + i\varepsilon)^2} + \frac{1}{P^+ - k^+}\frac{[2m\gamma^+ i\gamma_5 + 2k^+\sigma^{+-}i\gamma_5]}{k^2 - m^2 + i\varepsilon}\right\}u(P,S). \tag{3.3}$$

Most of the terms in Eq.(3.3) are harmless as $k^+ \to 0$. The only troublesome piece arises from the term proportional to k^-. In order to study its contribution further, one can make use of the algebraic identity

$$k^- = P^- - \frac{(\vec{P}_\perp - \vec{k}_\perp)^2}{2(P^+ - k^+)} - \frac{(P-k)^2}{2(P^+ - k^+)}. \tag{3.4}$$

One can easily verify that the first two terms on r.h.s. of Eq.(3.4) are well behaved as $k^+ \to 0$, when inserted in Eq.(3.3). The only troublemaker is the term proportional to $(P-k)^2$ in the numerator. Using

$$\int \frac{dk^-}{2\pi}\frac{1}{(k^2 - m^2 + i\varepsilon)^2} = \frac{i}{2}\frac{\delta(k^+)}{\vec{k}_\perp^2 + m^2}, \tag{3.5}$$

one finds for the singular piece

$$\begin{aligned} h_L^{sing}(x)MS^+ &= g^2 mS^+P^+\int \frac{dk^+d^2k_\perp}{(2\pi)^3}\frac{\delta(k^+ - xP^+)\delta(k^+)}{\vec{k}_\perp^2 + m^2} \\ &= g^2\frac{mS^+}{\pi}\delta(x)\int \frac{d^2k_\perp}{(2\pi)^2}\frac{1}{\vec{k}_\perp^2 + m^2} \\ &= g^2\frac{mS^+}{4\pi^2}\delta(x)\log\frac{\Lambda_\perp^2}{m^2} \end{aligned} \tag{3.6}$$

where $\Lambda_\perp^2$ is some cutoff and where we used $S\cdot P = 0$, i.e. $S^- = -S^+M^2/2P^+$. The factor S^+ on the r.h.s. arises from the matrix element of $i\gamma^+\gamma_5$. The precise numerical result is not so important here. What is important is that straightforward evaluation of our perturbative model yields a term proportional to a δ function for $h_L(x)$.

The singular term derived here is explicitly proportional to the current quark mass, i.e. only for strange or heavier quarks one expects a sizable contribution, and thus violation of the h_2-sum rule. However, it is not known whether this is only a lowest order artifact or whether this is a more general result. Certainly, using hard-soft factorization, this result implies a $\delta(x)$ term with nonzero coefficient also in the full (i.e. nonperturbative) result for $h_L(x)$ and thus the h_L-sum rule should be violated for strange and heavier quarks.

It should be emphasized that while we have identified the quark mass as one source of violation for the h_L-sum rule, it is by no means clear whether or not there are are other sources of further violation which would also affect light u and d quarks.

Before we are going to investigate the physical origin of the δ-function, let us first take a look at QCD perturbative corrections to $g_T(x)$ for a quark.

$$g_T(x)S_T^x = -g^2\bar{u}(P,S)\int \frac{d^4k}{(2\pi)^4}\delta(k^+ - xp^+)\gamma^\mu \frac{i}{\not{k} - m + i\varepsilon}\gamma^x i\gamma_5 \frac{i}{\not{k} - m + i\varepsilon}\gamma^\mu u(P,S)D_{\mu\nu}(P-k). \quad (3.7)$$

From the purely formal point-of-view the δ-function in $h_L(x)$ arises because of the term proportional to the light-front energy $k^- \equiv (k^0 - k^3)/\sqrt{2}$ in the numerator, which cancels the energy dominator for the gluon propagator. Armed with this insight, we can immediately focus on the potentially dangerous terms in g_T arising from the k^- in the numerator of the fermion propagator[6]

$$\gamma_\mu \left[k^-\gamma^+\gamma^x i\gamma_5 m + m\gamma^x i\gamma_5 k^-\gamma^+\right]\gamma^\mu = 0, \qquad (3.8)$$

i.e. because of the Dirac algebra, the coefficient of the potentially singular term in $g_T(x)$ vanishes. In contrast to the result for $h_L(x)$, at least in one loop order, radiative QCD corrections do not give rise to δ-functions in $g_T(x)$.

IV. LONG-RANGE CORRELATIONS ALONG THE LIGHT-CONE IN A SIMPLE TOY MODEL

In the previous section, we have seen that sum rules for higher-twist parton distributions can be easily violated. However, so far, we lack an intuitive understanding of the effect. In this (and the following) section we will turn our attention to simple toy models, where the same effect happens, but where it is easier to analyse. For the same reason, we will study the (spin independent) scalar higher-twist distribution $e(x)$ in the rest of the paper. Both, working in 1+1 dimensions and studying scalar distributions will simplify the algebra involved considerably and thus allow us to shed some light on the essential physics. The general conclusions should not be affected by these simplifications. One of the most simple examples for violations of sum rules for higher-twist distributions is a (1+1)-dimensional model of the Gross–Neveu type [14,15]

$$\mathcal{L} = \bar{u}_i\left(i\not{\partial} - m_0\right)u_i + \bar{s}_i\left(i\not{\partial} - m_0\right)s_i - \frac{\lambda}{N_c}\bar{u}_i u_i \bar{s}_j s_j \quad , \qquad (4.1)$$

where $i, j = 1, ..., N_C$ are "color" indices. Here we will restrict ourselves to leading order in $1/N_c$. Suppose we probe a u-quark with an external vector current which couples only to strange quarks. From the hadronic tensor

$$T^{\mu\nu}_{s/u}\left(q^2, p\cdot q\right) = i\int d^2x\, e^{iqx}\langle u,p\left|T\left(\bar{s}\gamma^\mu s(x)\bar{s}\gamma^\nu s(0)\right)\right|u,p\rangle \qquad (4.2)$$

we will consider only the symmetric part of the "+−" component. A typical Feynman diagram contributing to Eq. (4.2) to $\mathcal{O}(N_c^0)$ (leading order) is depicted in Fig. 2.

FIG. 2. Leading order $1/N_C$ contribution to $T^{\mu\nu}_{s/u}$ in the Gross–Neveu model. The dotted lines indicate the flow of momentum through the external currents $\bar{s}\gamma^\mu s$.

The OPE yields for $Q^2 = -q^2 \to \infty$

$$T_{s/u} \equiv \nu \cdot \frac{\left(T^{+-}_{s/u} + T^{-+}_{s/u}\right)}{2} = 2M^2\int_{-1}^{1} dx' \frac{x}{x^2 - x'^2} e_{s/u}(x') \qquad (4.3)$$

where

$$2M\, e_{s/u}(x) = \int_{-\infty}^{\infty} d\lambda\, e^{i\lambda x}\langle u,p\left|\bar{s}(0)s(\lambda n)\right|u,p\rangle \quad . \qquad (4.4)$$

Here M is the constituent quark mass [14,15]. Clearly, $e_{s/u}(x)$ satisfies a "sum rule" (the *sigma-term sum rule* [16])

$$\int_{-1}^{1} dx\, e_{s/u}(x) = \frac{1}{2M}\langle u\left|\bar{s}s\right|u\rangle \quad , \qquad (4.5)$$

which is non-zero and $\mathcal{O}(N_c^0)$ [15]. However, any attempt to verify Eq. (4.5) by studying physical cross sections is doomed to fail, since $\Im T^{\mu\nu}_{s/u} = 0$ to $\mathcal{O}(N_c^0)$.[7]

These seemingly contradictory results are consistent with dispersion theory. The crucial point is that the OPE result (4.3) is derived in a kinematic regime where $T^{\mu\nu}$ is purely real $|x| > 1$ and which is not accessible in DIS. A relation between T for $|x| > 1$ (where T is purely real) and $\Im T$ for $|x| < 1$ is established by means of a dispersion relation ($x > 1$)

[6]One can easily verify that terms with k^- in the numerator from the gauge field propagator are accompanied here by an explicit factor $\propto k^+$ in the fermion propagator and are thus harmless.

[7]This can be easily verified by applying the cutting rules to Fig. 1. Since $q^2 < 0$, at least one of the cut lines will have a negative invariant mass.

$$T_{s/u}(x) = p\left(\frac{1}{x}\right) + \frac{1}{\pi}\int_0^1 dx' \frac{x}{x^2 - x'^2} \Im T_{s/u}(x') \quad , \quad (4.6)$$

where $p\,(1/x)$ is an odd polynomial in $1/x$ which cannot be determined from dispersion theory — it corresponds to subtraction constants. If p would vanish, then a comparison between Eqs. (4.6) and (4.3) shows

$$\frac{2}{\pi}\Im T_{s/u}(x') = 2M^2 \left[e(x') + e(-x')\right] \quad . \quad (4.7)$$

However, if for example, $p(1/x) = c/x$ then

$$\frac{1}{\pi}\Im T_{s/u}(x') = c \cdot \delta(x') + 2M^2 \left[e(x') + e(-x')\right] \quad . \quad (4.8)$$

Since $\Im T$, which is measured experimentally, does not contain δ-functions, we expect them to be contained in $e(x') + e(-x')$. The latter will now be determined explicitly: obviously (Fig. 3)

$$\begin{aligned} &\langle u, p\,|\bar{s}(x)s(y)|\,u, p\rangle = \qquad (4.9)\\ &g^2 \int \frac{d^2k}{(2\pi)^2} \bar{u}(p)u(p) tr\left(\frac{e^{ikx}}{\not{k} - m + i\epsilon}\;\frac{e^{-iky}}{\not{k} - m + i\epsilon}\right) \quad , \end{aligned}$$

i.e.

$$e_{s/u}(x') = g^2 \int \frac{d\lambda}{2\pi} e^{i\lambda x'} \int \frac{dk^+\, dk^-}{(2\pi)^2} \frac{\left(k^2 + m^2\right) e^{-i\frac{k^+}{p^+}\cdot\lambda}}{\left(2k^+k^- - m^2 + i\epsilon\right)^2} \quad . \quad (4.10)$$

FIG. 3. Leading order $1/N_c$ Feynman diagram for Eq. (4.9).

Since the exponent in Eq. (4.10) does not contain k^-, one can always close the contour in the complex k^--plane such that no pole from the energy denominators is enclosed. For $k^+ \neq 0$ the surface term (from the semi-circle in the complex k^--plane) vanishes and one finds

$$\int dk^- \frac{k^2 + m^2}{\left(2k^+k^- - m^2 + i\epsilon\right)^2} \propto \delta(k^+) \qquad (4.11)$$

[The coefficient of proportionality is non-zero and calculable. It turns out to be such that $e_{s/u}(x)$ satisfies Eq. (4.5), as one can verify by comparison]. Therefore

$$e_{s/u}(x') \propto \delta(x') \quad . \qquad (4.12)$$

Mathematically the δ-function in Eq.(4.12) arises as a consequence of long-range correlations along the light-cone. In the Gross–Neveu model, to leading order in $1/N_c$, the light-cone correlator $\langle u, p\,|\bar{s}(0)s(x^-)|\,u, p\rangle$ is independent of x^- — which is a consequence of Eq. (2.11). Physically, this is related to the scalar coupling in the t-channel. This remark becomes more clear if we define the light-cone distributions by means of an infinite momentum boost [17].

Consider the equal time correlator ($p \equiv p^1$, $k \equiv k^1$)

$$\rho_p(k') = \int \frac{dx_1}{2\pi} e^{ikx_1} \langle u, p\,|\bar{s}(0)s(x_1)|\,u, p\rangle \quad , \qquad (4.13)$$

which measures nothing but the conventional (*i.e.* not light-cone) momentum distribution flowing through a strange current insertion (with scalar quantum numbers). It is related to the light-cone distribution through

$$e(x) = \lim_{p\to\infty} p \cdot \rho_p(x \cdot p) \quad . \qquad (4.14)$$

Now, as a consequence of the scalar t-channel coupling, $\rho_p(k')$ is actually independent of p. One finds

$$\begin{aligned} \rho_p(k) &\propto \int \frac{dk^0}{2\pi} tr\left(\frac{\not{k} + m}{k^2 - m^2 + i\epsilon}\right)^2 - {}''m \to M'' \\ &= \frac{k^2}{\left(k^2 + m^2\right)^{3/2}} - \frac{k^2}{\left(k^2 + M^2\right)^{3/2}} \qquad (4.15) \end{aligned}$$

where we subtracted the contribution from a heavy regulator, in order to render $\langle u\,|\bar{s}s|\,u\rangle = \int dk\, \rho_p(k)$ finite. Clearly, $\rho_p(k)$ (Eq. (4.15)) is a localized function with finite area. Measuring all momenta in units of p (which is after all what the area preserving mapping in Eq. (4.14) does) and performing the limit $p \to \infty$ thus gives rise to a distribution of zero width but finite area — a representation for a δ-function.[8]

The fact that $\rho_p(k)$ is independent of p, to leading order in $1/N$ is quite obvious, since the leading order in the $1/N$ approximation is similar to a mean field approximation. The expectation value of a scalar density, taken in a plane wave state, is a constant — independent of the momentum. Therefore, the u-quark gives rise to a source for s-quarks [through the point-like scalar coupling in Eq. (4.2)] which also does not depend on its momentum (4.15).

Thus one might suspect that the unique properties of the interaction term in Eq, (4.2) are the reason for our "weird" results:

[8] Using various test functions can can verify that Eq. (4.14) yields a "pure" δ-function and is not contaminated with higher multipoles, like δ''.

- $\Im T$ smooth (actually identically zero in this example);
- $\Im T$ violates the sum rule (the sum rule converges without problems);
- $e(x)$ contains a δ-function; and
- $e(x)$ satisfies the sum rule.

In particular, one might suspect that "more realistic" field theories (like QCD_{3+1}) behave differently. In the rest of this article we will demonstrate that some of these properties are actually quite generic for higher-twist distributions in many field theories (also QCD_{3+1}).

A non-trivial and non-perturbative example has already been given in Ref. [13], where the twist-4 parton distribution f_4 was analyzed in the context of QCD_{1+1} ($N_c \to \infty$). There, $\Im T^{--}$ (which scales against f_4 for non-zero x in the Bjorken limit) violates the sum rule which is derived for f_4. Furthermore, f_4 contains a δ-function, while $\lim_{\mathrm{Bj}} \Im T^{--}$ has only a mild (integrable) singularity for $x \to 0$.

In the above discussion, the presence of ultraviolet divergences in the Gross–Neveu model forced us to introduce a regulator (cutoff). In order to keep the discussion simple the issue of renormalization (and elimination of the cutoff dependence) has been avoided so far. This will be done in the following. To leading order in $1/N_c$ one still finds $\Im T_{s/u} = 0$. for the real part of the renormalized amplitude one finds [14], (Fig.2)

$$T^{\mu\nu}_{s/u}\left(q^2,\nu\right) = \tag{4.16}$$
$$g^2_{\sigma q\bar{q}} D_\sigma(0) \int \frac{d^2k}{(2\pi)^2} tr\left[\left(\frac{1}{\not{k} - M_F}\right)^2 \gamma^\mu \frac{1}{\not{k} + \not{q} - M_F}\gamma^\nu\right] \quad ,$$

where M_F is the physical fermion mass. $g_{\sigma q\bar{q}}$ is the physical quark-σ-meson coupling and $D_\sigma(0)$ is the σ-meson propagator evaluated at $p^2 = 0$ (note that no momentum flows through the σ-meson, since we are considering a forward amplitude). Therefore,

$$\begin{aligned} T_{s/u}\left(q^2,\nu\right) &= \nu\left(T^{+-}_{s/u} + T^{-+}_{s/u}\right) \\ &= c\frac{M_F^2 \nu}{Q^2}\int_0^1 dx \frac{\left(Q^2\right)^2 x(1-x)}{\left[Q^2 x(1-x) + M_F^2\right]^2} \\ &= c \cdot \frac{\nu \cdot M_F^2}{Q^2} \cdot f\left(\frac{Q^2}{M_F^2}\right) \quad , \end{aligned} \tag{4.17}$$

where c is some numerical constant. If one now introduces a Q^2-dependent parton distribution $e_{S/U}(x', Q^2)$ via the moment expansion

$$T_{s/u}(x,Q^2) = 2M_F^2 \sum_{\nu=0,2,4,\ldots} \frac{1}{x^{\nu+1}} \int_{-1}^1 dx' {x'}^2 e_{s/u}(x',Q^2) \tag{4.18}$$

one finds the unique solution for the even part of $e_{s/u}(x', Q^2)$

$$\begin{aligned} &\frac{e_{s/u}\left(x',Q^2\right) + e_{s/u}\left(-x',Q^2\right)}{2} = \frac{c}{4M_F} \cdot \delta(x') \cdot f\left(\frac{Q^2}{M_F^2}\right) \\ &\stackrel{Q^2\to\infty}{\longrightarrow} \frac{c}{2M_F}\delta(x') \cdot \log\frac{Q^2}{M_F^2} \quad . \end{aligned} \tag{4.19}$$

Since $T^{\mu\nu}$ is independent of ν, for all Q^2, one has $\nu \cdot T \propto 1/x (x = Q^2/2M_F\nu)$. Therefore, the δ-function is present for all values of Q^2. The coefficient multiplying $\delta(x')$ depends on $\log Q^2/M_F^2$ which reflects the running of the coupling constant in the Gross–Neveu model.

V. BOOSTING SLOWLY TO INFINITE MOMENTUM

In this Section we will continue our analysis of higher-twist distributions in various field theories, although we will now use perturbation theory to the lowest non-trivial order only. Of course, parton distributions in QCD cannot be calculated perturbatively, but this is not what we attempt to do here. The main motivation for this perturbative study is to shed some light on possible mechanisms for the violation of un-subtracted sum rules. The use of perturbation theory will allow us to explicitly compute various scattering amplitudes and correlations analytically. This is very important if one wants to understand the reason for the failure of the sum rules mathematically as well as physically.

The example we will be studying at is again a 1 + 1-dimensional model: massive fermions coupled to massive bosons with a chirally invariant Yukawa coupling (linear σ-model)

$$\begin{aligned} \mathcal{L} = \bar{\psi}\left(i\not{\partial} - m\right)\psi - \sigma\left(\Box + m^2\right)\sigma - \pi\left(\Box + m^2\right)\pi \\ + \gamma\bar{\psi}\left(\sigma + i\gamma_5\pi\right)\psi \quad , \end{aligned} \tag{5.1}$$

where, for simplicity, all masses are taken to be equal. The flavor structure [which is suppressed in Eq. (5.1)] is assumed to be such that only Fig. 4 contributes to order γ^2 to the hadronic tensor

$$T^{\mu\nu} \equiv i\int d^2x\, e^{iqx} \langle u,p\left|T\left\{\bar{d}\gamma^\mu s(0)\bar{s}\gamma^\mu d(x)\right\}\right|u,p\rangle \quad . \tag{5.2}$$

This can, for example, be accomplished by having only the flavor combination $\bar{u}d$ and $\bar{d}u$ couple to the mesons. Straightforward application of Feynman rules yields

one finds

$$\frac{1}{2}\left(T^{+-}+T^{-+}\right) = -4im^2\gamma^2\bar{u}(p)\int\frac{d^2k}{(2\pi)^2}\frac{\not{k}}{\left(k^2-m^2+i\epsilon\right)^2\left((k+q)^2-m^2+i\epsilon\right)\left[(p-k)^2-m^2+i\epsilon\right]}u(p)$$

$$= \frac{16m^2\gamma^2}{4\pi}\int_0^1 dy\int_0^{1-y}dz\frac{\left(zm^2-y\,p\cdot q\right)(1-y-z)}{\left[y(1-y)q^2+z(1-z)p^2+2yz\,p\cdot q-m^2+i\epsilon\right]^3} \quad . \tag{5.3}$$

For the application of dispersion relations it is helpful to perform an integration by parts in Eq. (5.3)

$$\frac{1}{2}\left(T^{+-}+T^{-+}\right) = \frac{m^2\gamma^2}{\pi}\int_0^1 dy\frac{(1-y)}{\left[y(1-y)q^2-m^2\right]^2} + \frac{m^2\gamma^2}{\pi}\int_0^1 dy\int_0^{1-y}dz\left\{\frac{2m^2(1-y-z)}{D^3}+\frac{1}{D^2}\right\} \quad , \tag{5.4}$$

where

$$D = y(1-y)q^2+z(1-z)p^2+2yz\,p\cdot q-m^2+i\epsilon \quad . \tag{5.5}$$

FIG. 4. Box diagram contribution to $T^{\mu\nu}$ (5.2) in the linear σ -model [Eq. (5.12)].

Most importantly, the first term in Eq. (5.4), which is purely real for $q^2<0$, does not depend on $p\cdot q$. The presence of this term thus ruins the application of an unsubtracted dispersion relation to $\left(T^{+-}+T^{-+}\right)/2$. As explained in Section II, the presence of subtraction constants in the dispersion relation is correlated with the presence of a δ-function in the light-cone momentum distribution (2.6):

With

$$2\,e_{d/u}(x) = i\gamma^2\int\frac{d^2k}{(2\pi)^2}\;\frac{\bar{u}(p)\;\not{k}\,\delta\left(\frac{k^+}{p^+}-x\right)u(p)}{\left(k^2-m^2+i\epsilon\right)^2\left((p-k)^2-m^2+i\epsilon\right)} \tag{5.6}$$

and

$$\not{k} \equiv k^+\gamma^-+k^-\gamma^+ = k^+\gamma^- + \left(p^- - \frac{m^2}{2\left(p^+-k^+\right)}\right)\gamma^+ - \frac{(p-k)^2-m^2}{2\left(p^+-k^+\right)}\gamma^+ \tag{5.7}$$

$$2m\,e_{d/u}(x) = -2\frac{m^2\gamma^2}{4\pi}\frac{\left(x+1-\frac{1}{1-x}\right)(1-x)\theta(x)\theta(1-x)}{\left[m^2x(1-x)-m^2\right]^2} + \frac{2m\gamma^2}{4\pi}\;\frac{\delta(x)}{m^2}$$
$$= \frac{\gamma^2}{2\pi m}\left\{\frac{x^2\theta(x)\theta(1-x)}{\left[x(1-x)-1\right]^2}-\delta(x)\right\} \quad , \tag{5.8}$$

where the δ-function came from the last term in Eq. (5.5). It should be emphasized here that the δ-function in $e(x)$ is closely related to the non-covariant counterterms in the Hamiltonian formulation of light-cone field theories [18–20]. It should also be emphasized that the same trick works in $3+1$-dimensions, *i.e.* there is nothing special about two dimensions here.
Mathematically, the δ-function appeared here because [18,19]

$$\int\frac{dk^-}{2\pi}\;\frac{1}{\left(k^2-m^2+i\epsilon\right)^2} = \delta(k^+)\int\frac{d^2k}{2\pi}\;\frac{1}{\left(k^2-m^2+i\epsilon\right)^2} = \frac{i}{2m^2}\delta(k^+) \quad . \tag{5.9}$$

From the physics point of view it is very instructive to consider a careful boost to infinite momentum. In analogy to Eq. (4.13) we consider the momentum distributions of d-quarks, in the wave function of a dressed u-quark of momentum p. One finds for the interaction in Eq. (5.1)

$$\rho_p(k) = \frac{m\gamma^2}{2\pi}\left\{-\frac{1}{\left(k^2+m^2\right)^{3/2}} + \int_0^1 dy\frac{(k-py)^2+m^2\left(y^2+\frac{1}{2}y-\frac{1}{2}\right)}{\left[(k-py)^2-y(1-y)m^2+m^2\right]^{5/2}}\right\} \quad . \tag{5.10}$$

Although there is no elementary scalar exchanged in the t-channel, $\rho_p(k)$ contains a term that does not depend on p at all. Upon substituting $k\to xp$ and performing the infinite momentum boost, this term piles up to a δ-function (again one verifies that no δ'' components are

present) as shown in Fig. 5. Actually $\lim\limits_{p\to\infty} p\rho_p(xp)$ gives a result identical to the light-cone calculation (5.8).

FIG. 5. The momentum distribution of d-quarks in a moving u-quark (5.8) for various values of the u-quark momentum. $p \cdot \rho_p(x)$ is plotted in units of $\gamma^2/\pi m$.

Although the linear σ-model does not yield a point-like scalar coupling in the t-channel, we observe a similar effect as in the GN-model: a part of the sea quark wave function does not follow the boosted valence component. In coordinate space language this means that only part of the sea quark distribution becomes Lorentz contracted when the u-quark is boosted, while another component looks the same in all frames! In 3+1 dimensional examples (similar to the one above) this means that dressed particles do not necessarily become "pancakes" as $p \to \infty$ and some component of the wavefunction retains a finite longitudinal extension. Whether that component is "visible" in some experiment depends of course on the quantum numbers of the probe and, in a manner which is not yet understood completely, on the details of the underlying theory.

If one calculates only

$$\Im T \equiv \frac{2pq}{\pi m} \Im \frac{T^{+-}+T^{-+}}{2} \quad , \tag{5.11}$$

(Fig. 6), one finds no indication whatsoever about the presence of a δ-function in $e(x)$. For all values of Q^2, $\Im T$ vanishes smoothly as $x_{\rm Bj} = Q^2/2p\cdot q \to 0$.[9] Also the Q^2 evolution does not rise to any "peak at small x."

[9] The divergence for large x in Fig. 6 is a purely $1+1$-dimensional artifact and arises from phase space factors.

FIG. 6. $p \cdot q \Im \left(T^{+-}+T^{-+}\right)/2$ (in units) as a function of $x = Q^2/2pq$ for various values of Q^2. Note the perfectly smooth behavior near $x = 0$.

Furthermore, as $Q^2 \to \infty$, $\Im T$ approaches $e(x)$ (Fig. 4) as long as $x \neq 0$. However, even though $\Im T$ is very smooth as $x \to 0$, the naive version of

$$2m \int dx\, e_{d/u}(x) = \langle u \left| \bar{d} d \right| u \rangle \quad , \tag{5.12}$$

where one replaces $2m\, e_{d/u}(x)$ by fails. This should be obvious since $\langle u \left|\bar{d}d\right| u\rangle$ is negative to order γ^2 in the above model, while $\Im(T)$ is always non-negative (from unitarity).

A different perspective is reached by computing the real part of T as well. In the Bjorken limit one finds

$$\begin{aligned} \Im T(x) &= \frac{\gamma^2}{2} x^2 \frac{\theta(x)\theta(1-x)}{\left[x(1-x)-1\right]^2} \\ \Re T(x) &= \frac{1}{\pi}\int_0^1 dx' \frac{\Im T(x')}{x-x'} - \frac{\gamma^2}{2\pi x} \qquad (x>1) \end{aligned} \tag{5.13}$$

which is consistent with the result from the OPE.[10]

$$\Re T(x) = m \int_{-1}^{1} dx' \frac{e_{d/u}(x')}{x-x'} \quad . \tag{5.14}$$

However, the presence of the subtraction term in (5.13) means that one can identify $\Im T(x')$ with $e(x')$ only for $x' \neq 0$.

It is instructive to re-analyze the situation from the point of view of the infinite momentum frame.[11] As

[10] Notice that, since the currents in Eq. (3.2) are not charge conjugation eigenstates, $e(x)$ and $T(x)$ do not have a definite transformation property under $x \to -x$ either.

[11] Of course, by making p in Eq. (3.8) large, one boosts the target and not the observer. However, this is equivalent.

$p \to \infty$ one almost recovers the naive picture, where all virtual constituents carry a positive, finite fraction of the hadrons' momentum: $p\rho(xp)$ drops sharply at $x = 1$ for $p = 10^3$ in Fig. 4 and vanishes for $x < 0$. This is consistent with the absence of vacuum fluctuations in the ∞-momentum frame because this would require at least one particle carrying a negative momentum fraction [17]. However, some component of the wave function of the u-quark "lacks behind" in the boost (5.10). This can be summarized in the following physical picture. As $p \to \infty$ most of the soft modes decouple from the system, which is partly responsible for the simplified dynamics in the *infinite momentum frame*. Of course these modes don't completely disappear but get concentrated in the region near $x = 0$. While leading twist distributions couple only weakly to this component of the wave function, there is a stronger coupling of the higher-twist distributions. That is why leading twist distributions do not contain these δ-functions. A very similar and related effect was first observed in the context of QED_{3+1} [21]. There it was shown that certain connected vacuum graphs do survive the $p \to \infty$ limit. Since $e(x)$ is, up to the momentum projection, proportional to a mass insertion, it should be clear that the non-covariant piece in the self-energy of an electron in QED — which is all that survives from the z-graph as $p \to \infty$ — is (up to mass derivative) proportional to the δ-function contribution one finds for $e(x)$ in QED.

I should emphasize again that, if such a situation applies to experimentally measured higher twist distributions, one would not be able to "see" any hint about the singular behavior of the light-cone distribution by just looking at the structure function.

VI. SUMMARY

We have investigated higher-twist distributions in a variety of field theories. This included perturbative examples in $1 + 1$ as well and $3 + 1$ dimensions.[12] In most cases the naive sum rules for the lowest moments of higher-twist distributions are violated when summed over the "experimentally determined"[13] distributions. For non-zero x_{Bj}, these structure functions scale towards the Fourier transformed quark-quark correlation in the hadron along the light-cone. The phenomenon which causes the failure of the sum rules for higher-twist structure functions is a δ-function at the origin in the light-cone distribution. Structure function and light-cone distribution coincide for non-zero x, while the δ-function is absent in the structure function. The sum-rules are valid when one includes the δ-function, *i.e.* when applied to the light-cone distribution. The structure functions then cease to satisfy the sum rule because they do not contain the δ-function. In that sense the δ-function destroys the sum rule. From a theorist's point of view, the δ-function restores it, *i.e.* when added to the structure function the combined result should reproduce the sum rule. The last point may however be of purely academic interest, since one cannot measure the light-cone correlation directly.

The relation between light-cone correlations and structure functions is usually based on the operator product expansion combined with dispersion relations. Unless one has a theoretical or experimental constraints on the high-energy behavior of the real part of the scattering amplitude, subtraction constants cannot be ruled out. In fact, the coefficient of the δ-function is proportional to such a high-energy subtraction. This also shows that the long-range correlation along the light-cone are consistent with the OPE.

In coordinate space language this means that only part of the sea quark distribution gets Lorentz contracted when the u-quark is boosted (giving rise to the finite-x_{Bj} component), while another component looks the same in all frames (contributing only at $x_{Bj} = 0$). In $3 + 1$-dimensional examples (similar to the one above) this means that the hadron does not become a "pancake" as $p \to \infty$. Only part of the wave function is Lorentz contracted while some rest retains its spherical shape.

There is nothing wrong or inconsistent about this result. It is only another example where naive intuition (in this case based on the Lorentz transformation properties in free field theory) fails. Interacting fields may have very complicated transformation properties which, in the above example, cause that part of the wavefunction does not Lorentz contract in the same way as free fields do. This also "explains" why higher-twist distributions are more vulnerable to these effects, since, in the parton model, they contain interaction terms explicitly.

From the point of view of null plane quantization, the δ-functions are a manifestation of zero-modes [24–30] in the hadronic wave function (see Appendix A). Higher-twist parton distributions are defined through correlations which involve bad currents (*i.e.* $\psi^{(-)}$ component of the spinors). When solving the constraint equation for $\psi^{(-)}$ one has to specify boundary conditions at $x^- = \pm\infty$. Non-zero values at the boundary are of course related to δ-functions in the Fourier transform. Another (not necessarily independent) connection to the failure of canonical light-cone quantization (without zero modes) is observed by considering the singular part of $e(x)$. In perturbation theory one finds that the derivative of the non-covariant γ^+/p^+ term in the fermion self-energy with respect to the quark mass is proportional to the δ-function coefficient in $e(x)$. Again this should not be a surprise,

[12] For a non-perturbative example in $1 + 1$ dimensions, see Ref. [22] for a discussion of f_4 in the context of QCD_{1+1} ($N_c \to \infty$). See also Ref. [23] for a discussion of the scalar density in the sine-Gordon model.

[13] Actually, the imaginary part of some scattering amplitudes. Of course, there can only be Gedanken experiments in model field theories.

since those terms are caused by an improper treatment of the zero modes [19] in canonical light-cone perturbation theory [31] — unless one uses several Pauli–Villars regulators (more than in a manifestly covariant approach) [19,32].

In practical applications, most sum rules for higher-twist distributions are probably useless since one expects them to diverge [33,34]. The discussion of $e(x)$ in this work should therefore be considered only a pedagogical example. One exception seems to be g_2. From a Regge pole analysis one expects the Burkhardt–Cottingham sum rule to converge [4]. Also, as shown in Section III, perturbation theory does not indicate any divergences in QCD. Furthermore, Regge pole analysis [4] as well as perturbative considerations (Section IV) suggest validity of the sum rule. However, no strict predictions for g_2 have been made on a non-perturbative basis so far and at this point one has to wait for the experimental result to see whether the g_2-sum rule is valid.

For strange or heavier quarks, the h_L sum rule seems to be most certainly violated. The perturbative analysis in this work only yielded a violation proportional to the quark mass and thus one might be tempted to assume validity of the h_L sum rule for u and d quarks. However, it is not clear whether other effects, which have not been considered in this work, lead to a violation for light quarks as well.

Even if these sum-rules are violated, the OPE remains valid. What needs to be added is a subtraction in the dispersion relation. Note that such a mechanism cannot be used to "explain" the spin crisis, since g_1 is leading twist and thus a subtraction in the dispersion relation would be in conflict with unitarity constraints.

APPENDIX A: $E(X)$ ON A FINITE INTERVAL

The δ-function at the origin in $e(x)$ strongly suggests a relation to zero modes in light-cone quantization [24,27–30]. Due to the severe infrared singularities in light-cone quantization (the kinetic energy diverges as $k^+ \to 0$) it turns out to be necessary to go on a finite interval. Furthermore, following Lenz *et al.* [24], we will use a tilted coordinate system

$$\hat{x}^- = x^- \quad , \qquad \hat{x}^+ = x^+ + \epsilon \frac{x^-}{L} \quad , \tag{A1}$$

where the $\hat{}$ refers to the "usual" light-cone coordinates

$$\hat{x}^+ = \left(x^0 \pm x^1\right)/\sqrt{2} \quad . \tag{A2}$$

The notation in this Appendix will thus be different from the rest of the paper in order to allow for a direct comparison with Ref. [24], which is strongly recommended to supplement this Appendix.

This implies for example for scalar products

$$a \cdot b = \hat{a}_+ \hat{b}_- + \hat{a}_- \hat{b}_+ = a_+ \left(b_- + \frac{\epsilon}{L} b_+\right) + b_+ \left(a_- + \frac{\epsilon}{L} a_+\right) \quad . \tag{A3}$$

L refers to the length of the interval and we will impose periodic boundary conditions in x^-. The canonically conjugate momentum p_- thus becomes quantized

$$p_{-n} = \frac{2n\,\pi}{L} \quad . \tag{A4}$$

For the linear σ-model (5.1) one thus finds to order g^2

$$\begin{aligned} e(k_-) &\propto \int dk_+ \frac{\not{k}}{\left(k^2 - m^2 + i\epsilon\right)^2} \, \frac{1}{(p-k)^2 - m^2 + i\epsilon} \\ &= \frac{c}{\sqrt{\epsilon \cdot L}} \left\{ \int_0^1 dx \left[\frac{1}{2D^{3/2}} - \frac{3}{4}\, \frac{m^2(1-x)}{D^{5/2}} \right] \right. \\ &\qquad \left. - \frac{1}{2} \left[\frac{L}{2\epsilon} k_-^2 + m^2 \right]^{-3/2} \right\} \quad , \end{aligned} \tag{A5}$$

where

$$D = \frac{L}{2\epsilon} \left(k_- - x p_-\right)^2 - x(1-x)m^2 + m^2 \quad . \tag{A6}$$

Of course k_- as well as p_- take on only discrete values (A.4). Here c is some numerical constant proportional to g^2. One immediately recognizes the similarity to Eq. (3.8), which is not completely accidental, since the infinite momentum boost also gives some kind of regularized representation for $e(x)$. The canonical light-cone limit for $e(k_-)$ is obtained by taking $\epsilon L m^2 \to 0$ while keeping n_k and n_p fixed ($k_- = 2\pi n_k/L$, $p_- = 2\pi n_p/L$). Using

$$\int_0^1 \frac{dx}{\sqrt{\epsilon L}} f\left(\frac{x-y}{\sqrt{\epsilon L}}\right) \stackrel{\sqrt{\varepsilon L} \to 0}{\longrightarrow} \int_{-\infty}^{\infty} dx\, f(x) \quad , \tag{A7}$$

this yields ($y = \frac{k_-}{p_-}$),

$$e(k_-) \to c \cdot \sqrt{2} \frac{y^2 m^2}{\left[y(1-y)m^2 - m^2\right]^2} \theta(y)\theta(1-y) \quad , \tag{A8}$$

for $n_k \neq 0$. However, for $n_k = 0$,

$$e(0) \to -\frac{c}{\sqrt{\epsilon L}} \cdot \frac{1}{m^3} \quad , \tag{A9}$$

i.e. the zero mode contribution diverges.

The mere fact that $e(0)$ diverges reflects the existence of the δ-function in the continuum calculation. However, if one actually wants to make a quantitative comparison with Section III, one has to perform a more sophisticated continuum limit where also the hadron momentum n_k (in discrete units) approaches infinity.

[1] A.J.G.Hey and J.E. Mandula, Phys. Rev. **D5**, 2610 (1972); K. Sasaki, Prog. Theor. Phys. **54**, 1816 (1975); M.A. Ahmed and G.G. Ross, Nucl. Phys. **B111**, 441 (1976).
[2] R.L. Jaffe and X. Ji, Phys. Rev. Lett. **67** 552 (1991); R.L. Jaffe and X. Ji, Nucl. Phys. **B375**, 527 (1992).
[3] HERMES collaboration. For details point WWW server to http://dxhra1.desy.de/documentation /technicalreport/index.html
[4] H. Burkhardt and W.N. Cottingham, Ann. Phys. (N.Y.) **56**, 453 (1970).
[5] M. Burkardt, Proc. of 10th International Symposium on High Energy Spin Physics, Nagoya, Japan, Nov.1992, Eds. Hasegawa et al. (Universal Academy Press, Tokyo, 1993).
[6] R.D. Tangerman and P.J. Mulders, hep-ph/9408305.
[7] RHIC spin collaboration, proposal 1992.
[8] R.L. Jaffe, Proc. of 10th International Symposium on High Energy Spin Physics, Nagoya, Japan, Nov.1992, Eds. Hasegawa et al. (Universal Academy Press, Tokyo, 1993).
[9] F.E. Close and D. Sivers, Phys. Rev. Lett. **39**, 1116 (1977); J.C. Collins and D.E. Soper, Phys. Rev. **D16**, 2219 (1977).
[10] J. Ralston and D.E. Soper, Nucl. Phys. **B152**, 109 (1979).
[11] T.-P. Cheng and L.-F. Li, *Gauge Theory and Elementary Particle Physics* (Oxford University Press, Oxford, 1984); T. Muta, *Foundations of QCD* (World Scientific, Singapore, 1987).
[12] R.L. Jaffe, Comm. Nucl. Part. Phys. **19**, 239 (1990).
[13] S.J. Brodsky, F.E. Close and J.F. Gunion, Phys. Rev. **D5**, 1384 (1972); *ibid.***D8**, 3678 (1973).
[14] D. J. Gross and A. Neveu, *Phys. Rev.* **D10** (1974) 3235; for a review, see *e.g.* B. Rosenstein, B. J. Warr and S. H. Park, *Phys. Rep.* **205**, 59 (1991).
[15] M. Burkardt, *Phys. Lett.* **B268**, 419 (1991).
[16] D.J. Broadhurst, J.F. Gunion and R.L. Jaffe, Phys. Rev. **D8** 566 (1973); R. L. Jaffe, Phys. Rev. **D21**, 3215 (1980).
[17] S. Weinberg, *Phys. Rev.* **150**, 1313 (1966).
[18] S. Chang and S. Ma, *Phys. Rev.* **180**, 1506 (1969); T. M. Yan, *Phys. Rev.* **D7**, 780 (1973).
[19] M. Burkardt and A. Langnau, Phys. Rev. **D44**, 1187; 3857 (1991).
[20] D. Mustaki, S. Pinsky, J. Shigemitsu and K. Wilson, Phys. Rev. **D43**, 3411 (1991).
[21] S. J. Brodsky, R. Roskies and R. Suaya, *Phys. Rev.* **D8**, 4574 (1973).
[22] M. Burkardt, Nucl. Phys. **B373**, 613 (1992).
[23] M. Burkardt, Phys. Rev. **D47**, 4628 (1993).
[24] F. Lenz, S. Levit, M. Thies and K. Yazaki, Ann. Phys.(N.Y.) **208**, 1 (1991); F. Lenz, Proceedings of NATO Advanced Study Institute on “Hadrons and Hadronic Matter”, eds. Vautherin et al. (Plenum, New York, 1990).
[25] K. Hornbostel, in “From Fundamental Fields to Nuclear Phenomena”, eds. J. A. McNeil and C. E. Price, (World Scientific, Singapore, 1991).
[26] K. Hornbostel, Phys. Rev. **D45**, 3781 (1992).
[27] Th. Heinzl, S. Krusche and E. Werner, *Phys. Lett.* **B256** (1991) 55; **B272**, 54 (1991); **B275**, 410 (1992).
[28] G. McCartor, *Z. Phys.* **C52**, 611 (1991).
[29] A. C. Kalloniatis and H. C. Pauli, Z. Phys.**C60**, 255 (1993); *ibid.* **C63**, 161 (1994). A. C. Kalloniatis and D. G. Robertson, Phys. Rev. **D50**, 5262 (1994).
[30] S. S. Pinsky and B. Van de Sande, Phys. Rev. **D48**,816 (1993); *ibid.* **D49**, 2001 (1994).
[31] S. J. Brodsky and G. P. Lepage, *Phys. Rev.* **D22**,2157 (1980).
[32] C. Bouchiat, P. Fayet and N. Sourlas, *Lett. Nouvo Cimento* **4**, 9 (1972); S.-J. Chang and T.-M. Yan, *Phys. Rev.* **D7**, 1147 (1983).
[33] D. A. Dicus, R. Jackiw and V. L. Teplitz, *Phys. Rev.* **D4**, 1733 (1971).
[34] R. Jackiw and R. E. Waltz, *Phys. Rev.* **D6**, 702 (1972; R. Jackiw, *Springer Tracts in Modern Physics*, vol. 2, G. Höhler, ed (Springer, Berlin, 1972), p. 1.